GG 611 Big Gulp
Fall 2014

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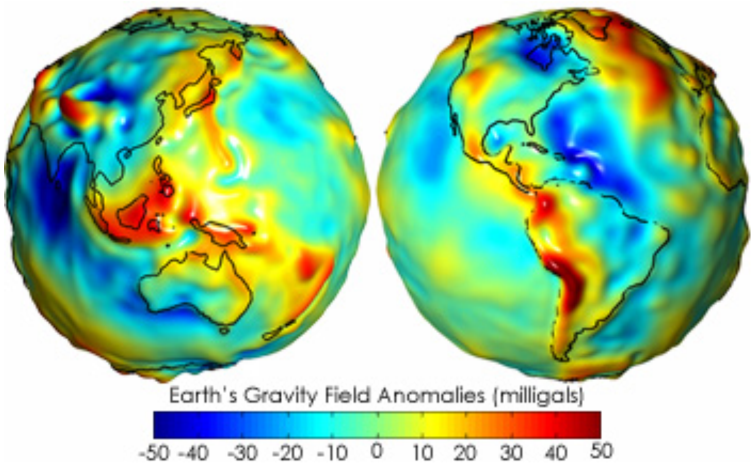
Gravity, the Geoid, and Mantle Dynamics

Lecture: Gravity and the Geoid



Gravitational Potential

For a point mass:

Newton's law of gravitation: $\vec{F} = m\vec{a} = -G\frac{mM}{r^2}$

Then the acceleration due to gravity is: $g = -G\frac{M}{r^2}\hat{r}$

The gravitational potential U_G is the potential energy per unit mass in a gravitational field. Thus:

$$m\mathrm{d}U_G = -F\mathrm{d}r = -mg\mathrm{d}r$$

Then the gravitational acceleration is: $\vec{g} = -\vec{\nabla}U = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)U$

The gravitational potential is given by: $U_G = -G\frac{M}{r}$

For a distribution of mass:
Everywhere outside a sphere of mass M: $U_G = -G\frac{M}{r}$

Centrifugal Potential

For a rotating body such as Earth, a portion of gravitational self-attraction drives a centripetal acceleration toward the center of the Earth. When viewed in the frame of the rotating body, the body experiences a centrifugal acceleration away from the Earth's axis of rotation.

Angular velocity: $\omega = \frac{d\theta}{dt} = \frac{v}{x}$ where $x = r\sin\theta$

Centrifugal acceleration: $a_c = \omega^2 x = \frac{v^2}{x}$

But $\vec{a}_c = -\vec{\nabla} U_c$, so we can calculate the centrifugal potential by integrating:

$U_c = -\frac{1}{2}\omega^2 x^2 = -\frac{1}{2}\omega^2 r^2 \sin^2\theta$

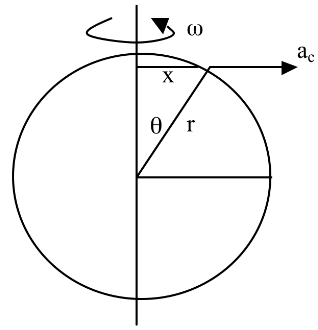
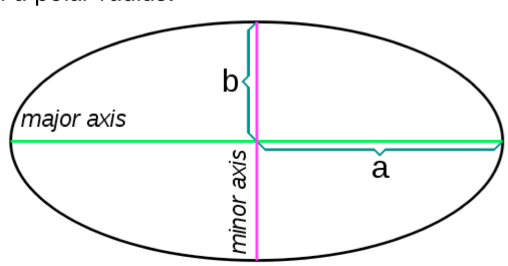


Figure of the Earth

Earth's actual surface is an equipotential surface (sea level), a surface for which $U_G + U_c = \text{constant}$. The figure of the Earth a smooth surface that approximates this shape and upon which more complicated topography can be represented. The earth approximates an oblate spheroid, which means it is elliptically-shaped with a longer equatorial radius than a polar radius.

The flattening (or oblateness) is the ratio of the difference in radii to the equatorial radius:

$f = \frac{a-b}{a}$



For earth, *f*=0.00335287, or 1/298.252, and the difference in the polar and equatorial radii is about 21 km.

The **International Reference Ellipsoid** is an ellipsoid with dimensions:

Equatorial Radius:	a = 6378.136 km
Polar Radius	c = 6356.751 km
Radius of Equivalent Sphere:	R = 6371.000 km
Flattening	f = 1/298.252
Acceleration Ratio	$m = \frac{a_C}{a_G} = \frac{\omega^2 a^3}{GM_E} = 1/288.901$
Moment of Inertia Ratio	$H = \frac{C-A}{C} = 1/305.457$

The gravitational potential of the Earth (the geopotential) is given by:

$$U_g = U_G - \frac{1}{2}\omega^2 r^2 \sin^2\theta = -\frac{GM}{r} + \frac{G}{r^3}(C-A)\left(\frac{3\cos^2\theta - 1}{2}\right) - \frac{1}{2}\omega^2 r^2 \sin^2\theta$$

where θ = colatitude (angle measured from the north pole, or 90-latitude).

The geopotential is a constant (U_0) everywhere on the reference ellipsoid.

Gravity on the Reference Ellipsoid

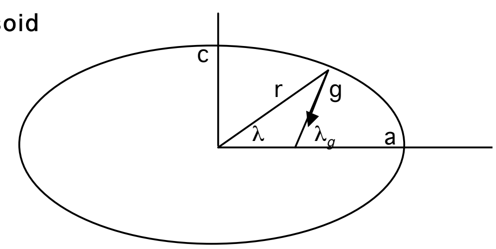
To first order: $r = a\left(1 - f\sin^2\lambda\right)$

Geocentric latitude = λ

(measured from center of mass)

Geographic latitude = λ_g

(in common use)



To first order: $\sin^2\lambda \approx \sin^2\lambda_g - f\sin^2 2\lambda_g$

The acceleration of gravity on the reference ellipsoid is given by: $\vec{g} = -\vec{\nabla}U_g$

Performing this differentiation gives:

$$g = 9.780327\left[1 + 0.0053024\sin^2\lambda_g + 0.0000059\sin^2 2\lambda_g\right]$$

Equatorial gravity is: g_e=9.780327 m/s^2

Polar gravity is: g_p=9.832186 m/s^2

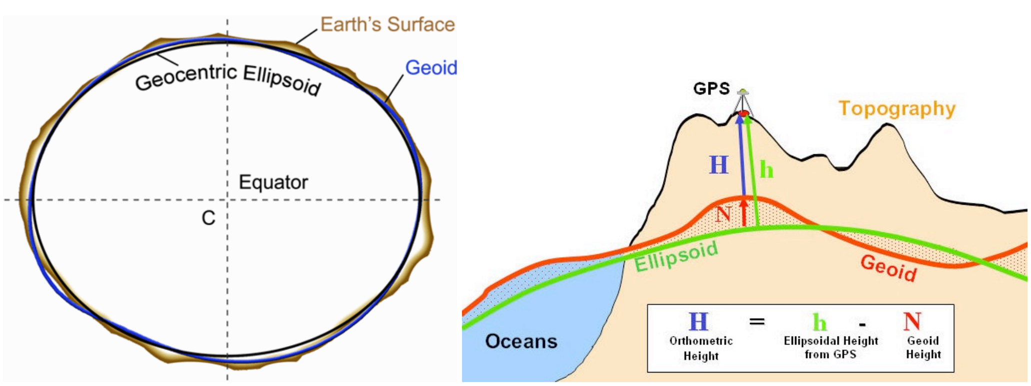
Gravity on the ellipsoid:

The poleward increase in gravity is 5186 mgal, and thus only about 0.5% of the absolute value.
(gravity is typically measured in units of mgal = 10^{-5} m/s^2).

Gravity decreases toward to pole because :
(1) The pole is closer to the center of Earth than the equator (~6600 mgal)
(2) The pole does not experience centrifugal acceleration (~3375 mgal)

These are countered by:
(3) The equator has more mass because of the bulge, which increases equatorial gravity (~4800 mgal)

Together these three affects yield the 5186 mgal difference.

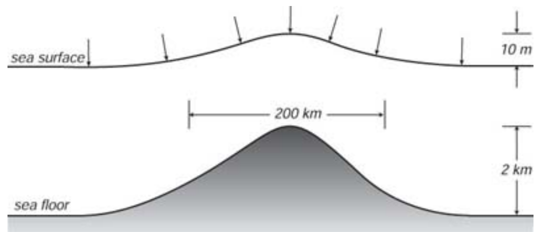


The **geoid** is the equipotential surface that defines sea level, and is expressed relative to the reference ellipsoid.

Deflections of the geoid away from the reference ellipsoid are caused by by lateral variations in the internal densities of the Earth.

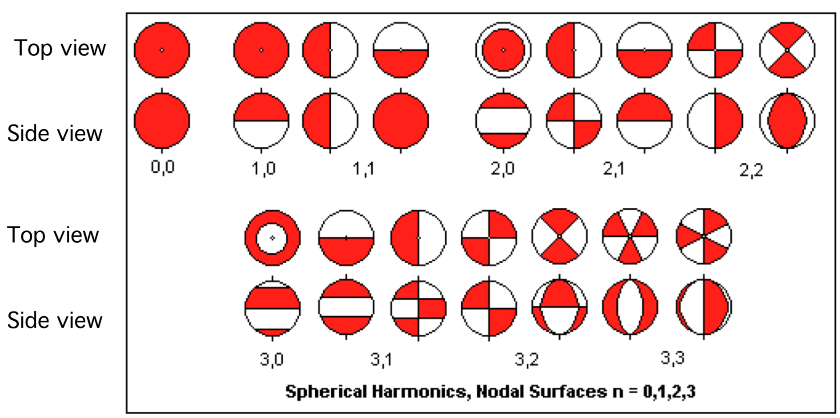
Temporal variations in the geoid are caused by changes in distribution of masses (primarily hydrological) upon the surface of the Earth.

Mass excess (either subsurface excess density or positive topography) deflects the geoid upwards.



Spherical Harmonics

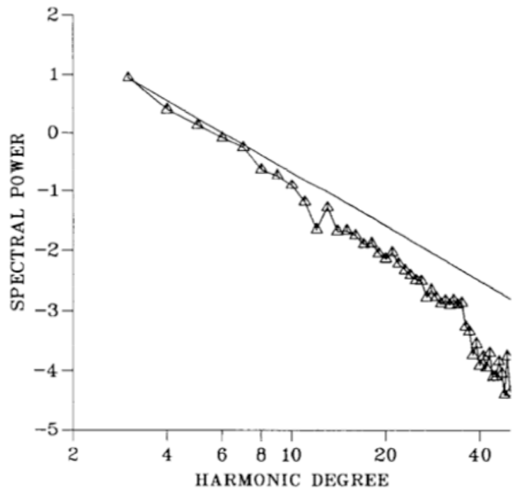
The geoid (and any function on a sphere) can be expressed in terms of spherical harmonics of degree *n* and order *m*: $Y_n^m = \left(a_n^m \cos m\phi + b_n^m \sin m\phi\right) P_n^m\left(\cos\theta\right)$

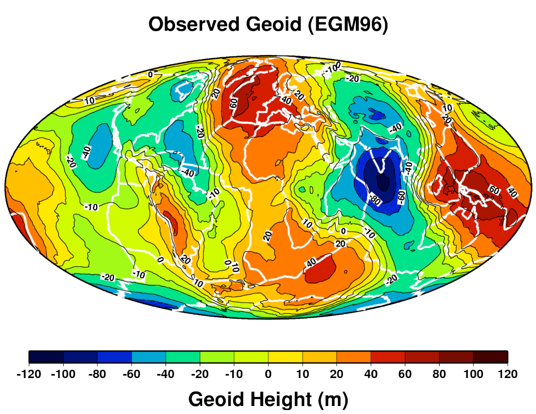


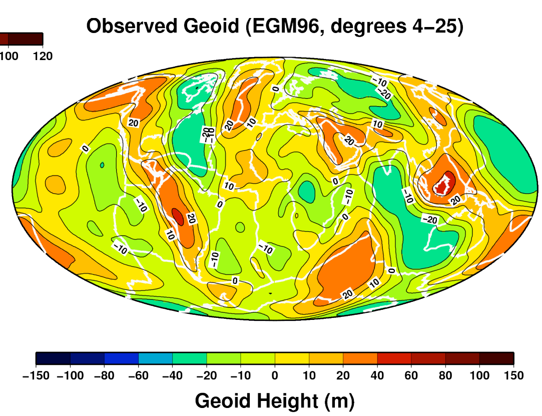
The power spectrum of the geoid is given by:

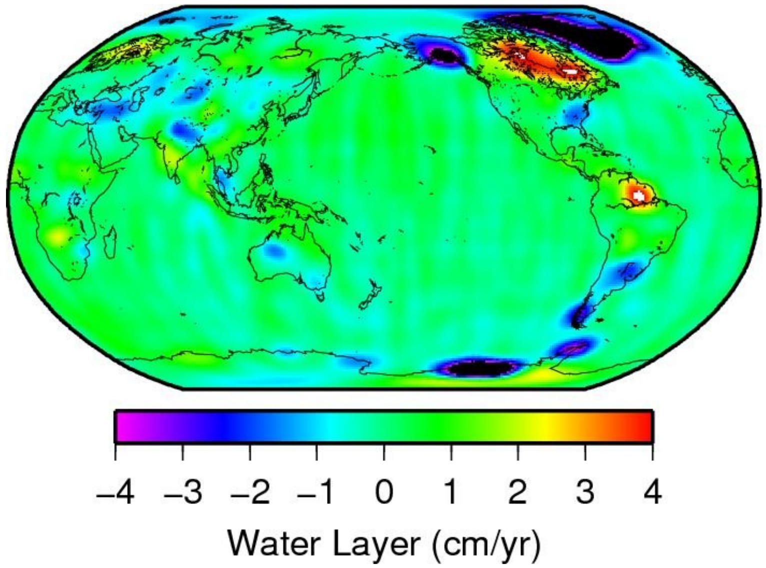
$$P_n = \sum_{m=0}^{n} \left(a_{nm}^2 + b_{nm}^2 \right)$$

The dominance of the low-harmonic degrees in the geoid power spectrum indicate that the dominant shape of the geoid is controlled by structures deep within the mantle.









Water mass changes inferred from geoid movements 2003-2009
(as measured from the GRACE satellite)

Measurement of Absolute Gravity:

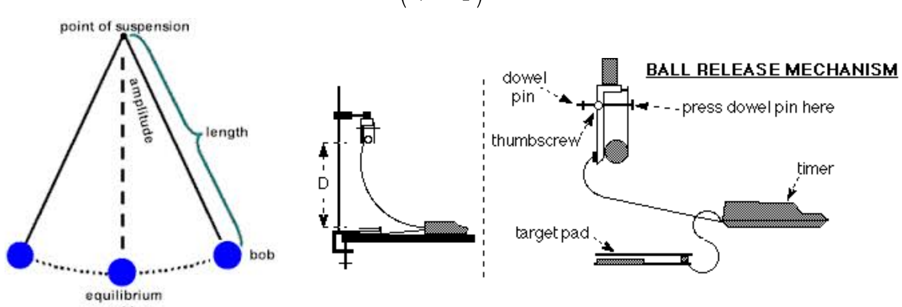
Pendulum Method: Measure the period $T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{L}{g}}$

To measure 1 mgal variation, the period must be measured to within 1μs.

Free-fall Method: Measure the fall of a mass: $z = z_0 + ut + gt^2/2$

To measure 1 μgal variation, time must be measured to within 1ns.

Rise-and-fall Method: Measure time T for a thrown ball to rise and fall a height z: $z = g(T/2)^2/2$. Then $g = \frac{8(z_1 - z_2)}{(T_1^2 - T_2^2)}$. μgal precision; not portable.



Measurement of Relative Gravity:

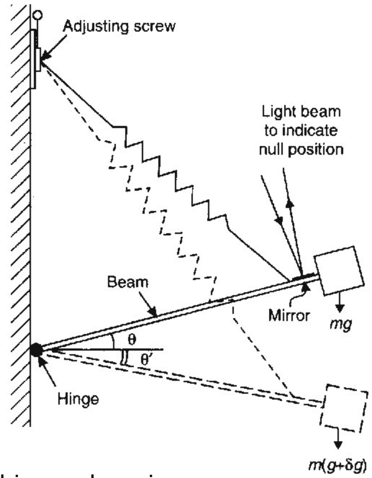
Stable Gravimeter: Measure Δs, the change in a spring's length: $\Delta g = \frac{k}{m}\Delta s$

Unstable Gravimeter: Use a spring with built-in tension, so: $\Delta g = \frac{k}{m} s$

(LaCost-Romberg gravimeter)

Usage: Adjust the spring length to zero using a calibrated screw.

Sensitivity: 0.01 mgal for a portable device.



Superconducting Gravimeter: Suspend a niobium sphere in a stable magnetic field of variable strength. Sensitivity: 1 ngal

Gravity Corrections:

Many lateral and temporal variations in gravity can be predicted, and thus removed from a gravity survey to isolate the "interesting" variations.

Drift Correction: In relative gravity surveys, instrument drift can be corrected by making periodic measurements at a base station with known gravity.

Tidal Correction: Gravity changes during the day due to the tides in a known way. Tidal corrections can be computed precisely if time is known. For example, if the moon is directly overhead, the tidal correction would be:

$$\Delta g_T = G\frac{M_L}{r_L^2}\left(\frac{2R_E}{r_L} + 3\left(\frac{R_E}{r_L}\right)^2 + \dots\right)$$

This should be added to measured gravity.

Eötvös Correction: Moving eastward at v_E, your angular velocity increases by: $\Delta\omega = v_E / (R_E \cos\lambda)$. This change increases the centrigugal acceleration:

$\Delta a_C = \left(\frac{da_C}{d\omega}\right)\Delta\omega = (2\omega R_E \cos\lambda)\left(\frac{v_E}{R_E \cos\lambda}\right) = 2\omega v_E$. Downward gravity changes by:

$\Delta g = -2\omega v_E \cos\lambda$. *The Eötvös effect decreases gravity when moving east.*

Gravity Corrections:
Many lateral and temporal variations in gravity can be predicted, and thus removed from a gravity survey to isolate the "interesting" variations.

Latitude Correction: Absolute gravity is corrected by subtracting normal gravity on the reference ellipsoid: $g_n = g_e\left(1+\beta_1 \sin^2\lambda + \beta_2 \sin^4 2\lambda\right)$

where $g_e = 9.780327$ m/s^2, $\beta_1 = 5.30244\times10^{-3}$, and $\beta_2 = -5.8\times10^{-6}$.

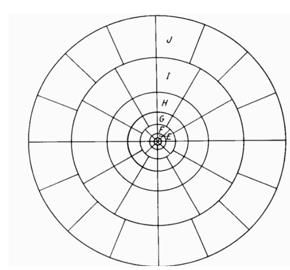
Relative gravity is corrected by differentiating g_n with respect to λ:

$\Delta g_{lat} = 0.8140 \sin 2\lambda$ mgal per km north-south displacement. *This correction is subtracted from stations closer to the pole than the base station.*

Gravity Corrections:
Many lateral and temporal variations in gravity can be predicted, and thus removed from a gravity survey to isolate the "interesting" variations.

Terrain Correction: Nearby topography perturbs gravity measurements upward due to mass mass excess above the station (nearby hills) or due to mass deficiency below the station (nearby valleys). The terrain correction is computed using:

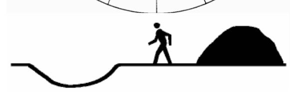
$\Delta g = G\left(\mathrm{d}m\cos\theta\right)/\left(r^2+z^2\right)$



where r and z are the horizontal and vertical distances to dm, and θ is the angle to the vertical. *The terrain correction is always positive.*

Integrating over a sector gives:

$\Delta g_T = G\rho\phi\left(\left(\sqrt{r^2+h^2}-r_1\right)-\left(\sqrt{r^2+h^2}-r_2\right)\right)$



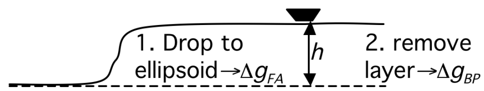
r_1 and r_2 are the inner and out radii, h is the height, ϕ is the sector angle.

Gravity Corrections:
Many lateral and temporal variations in gravity can be predicted, and thus removed from a gravity survey to isolate the "interesting" variations.

Bouguer Plate Correction: This correction compensates for a rock layer of thickness h between the measurement elevation level and the reference level. For a solid disk of density ρ and radius r, the terrain correction is: $\Delta g_T = 2\pi G\rho\left(h - \left(\sqrt{r^2 - h^2} - r\right)\right)$. Allowing r to become infinite, we obtain:

$\Delta g_{BP} = 2\pi G\rho h = 0.0419 \times 10^{-3}\rho$ mgal/m if ρ is in kg/m^3.

This correction must be subtracted, unless the station is below sea level in which case a layer of rock must be added to reach the reference level.



For gravity measured over water, water must be replaced with rock by assigning a slab with density $\left(\rho_{\text{rock}} - \rho_{\text{water}}\right)$.

Gravity Corrections:
Many lateral and temporal variations in gravity can be predicted, and thus removed from a gravity survey to isolate the "interesting" variations.

Free-air Correction: This correction compensates for gravity's decrease with distance from the Earth's surface. It is determined by differentiating g:

$$\Delta g_{FA} = \frac{\partial}{\partial r}\left(-G\frac{M_E}{r^2}\right) = +2G\frac{M_E}{r^3} = -\frac{2}{r}g = 0.3086 \text{ mgal/m}$$

This correction must be added (for stations above sea level).

Combined Correction: Free air and Bouguer corrections are often combined:

$$\Delta g_{FA} + \Delta g_{BP} = \left(0.3086 - 0.0419\rho \times 10^{-3}\right) \text{ mgal/m} = 0.197 \text{ mgal/m}$$

assuming a crustal density of 2670 kg/m^3. To obtain 0.01 mgal accuracy:

-- location must be known to within 10 m (for latitude correction)

-- elevation must be known to within 5 cm (for combined correction)

Gravity Anomalies

After the appropriate corrections are applied, gravity data reveal information subsurface density heterogeneity. How should they be interpreted?

Gravity over a Uniform Sphere

Gravity for a sphere is the same as for a point mass. The z-component:

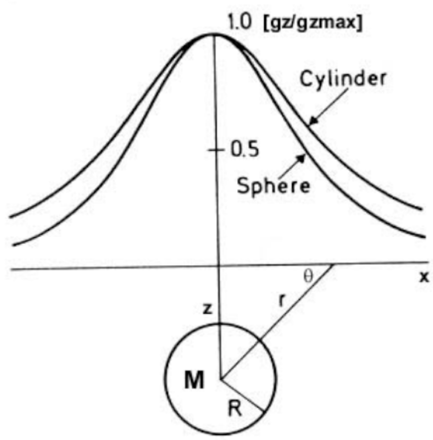
$$\Delta g_z = \Delta g \sin\theta = G\frac{M}{r^2}\frac{z}{r} \quad \text{where}$$

$$M = \frac{4\pi}{3}R^3\Delta\rho \quad \text{and} \quad r^2 = z^2 + x^2 \text{ giving:}$$

$$\Delta g_z = \frac{4\pi}{3}G\left(\frac{\Delta\rho R^3}{z^2}\right)\left(\frac{z^2}{z^2+x^2}\right)^{3/2}$$

The maximum is at x=0, where:

$$\Delta g_{zmax} = \frac{4\pi}{3}G\left(\frac{\Delta\rho R^3}{z^2}\right)$$



Rule of thumb: $z = 0.65w$ where w is the width at half height of the anomaly.

Gravity Anomalies

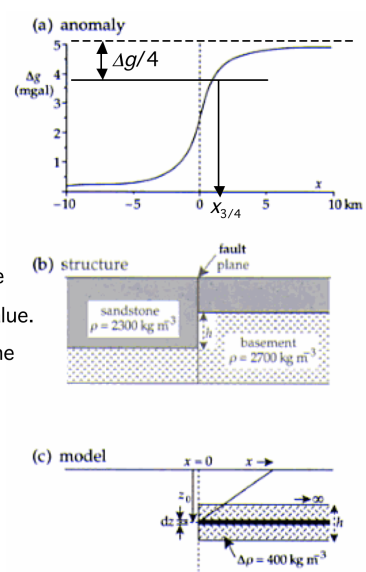
After the appropriate corrections are applied, gravity data reveal information subsurface density heterogeneity. How should they be interpreted?

Gravity over a Semi-Infinite Horizontal Sheet

A horizontally truncated thin sheet can be used to approximate a bedded formation offset by a fault. If the fault is centered at x=0, z_0=0, then the gravity anomaly is: $\Delta g_z = 2G\Delta\rho h\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{z_0}\right)\right)$

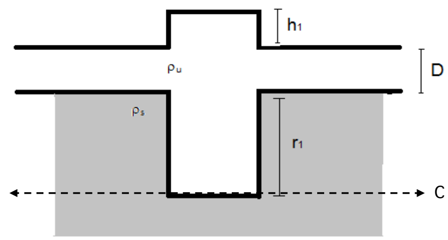
Rule of thumb: $z_0 \sim x_{1/4} \sim x_{3/4}$

Where $x_{1/4}$ and $x_{3/4}$ are the positions where the gravity anomaly is ¼ and ¾ its max value. Note that as x→∞, $\Delta g_z = 2G\Delta\rho h$, which is the solution for a Bouguer Plate anomaly.



Isostasy

Long wavelength variations in topography are isostatically compensated at depth. This means that the excess mass in positive topography is compensated by a mass deficiency at depth.



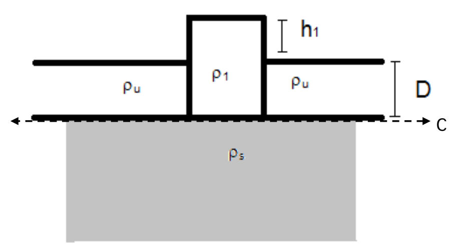
Airy Isostasy:

Lateral variations in crustal thickness allow surface topography to be compensated by a deep crustal root. The thickness of this root is determined by requiring the mass in columns above the compensation depth (C) to be equal:

$r_1 = \frac{\rho_c}{\rho_m - \rho_c} h_1$ or, if the topography is under water, $r_1 = \frac{\rho_c - \rho_w}{\rho_m - \rho_c} h_1$

Isostasy

Long wavelength variations in topography are isostatically compensated at depth. This means that the excess mass in positive topography is compensated by a mass deficiency at depth.



Pratt Isostasy:

Lateral variations in crustal density compensate topography, so again the mass in columns above the compensation depth (C) are equal. The density is:

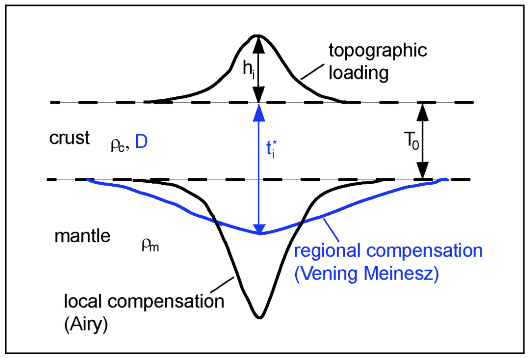
$\rho_1 = \frac{D}{h_1 + D}\rho_c$ or, if the column is a depth d under water, $\rho_0 = \frac{\rho_c D - \rho_w d}{D - d}$

Isostasy

Long wavelength variations in topography are isostatically compensated at depth. This means that the excess mass in positive topography is compensated by a mass deficiency at depth.

Vening Meinesz Isostasy:

In this type of isostasy, short-wavelength topography is supported by the elastic strength of the crustal rocks. The load is instead distributed by the bent plate over a broad area. This distributed load is compensated.



Gravity Anomalies over Topography

Uncompensated topography (Short-wavelengths)

Free-air anomaly (apply the free-air correction only):

$\Delta g+\Delta g_{FA} >> 0$ because of the topography's excess mass

Bouguer anomaly (apply both free-air and Bouguer plate corrections):

$\Delta g+\Delta g_{FA}-\Delta g_{BP} \sim 0$ because Bouguer corrects for excess mass.

Compensated topography (Long-wavelengths)

Free-air anomaly (apply the free-air correction only):

$\Delta g+\Delta g_{FA} \sim 0$ because topography is compensated (no excess mass)

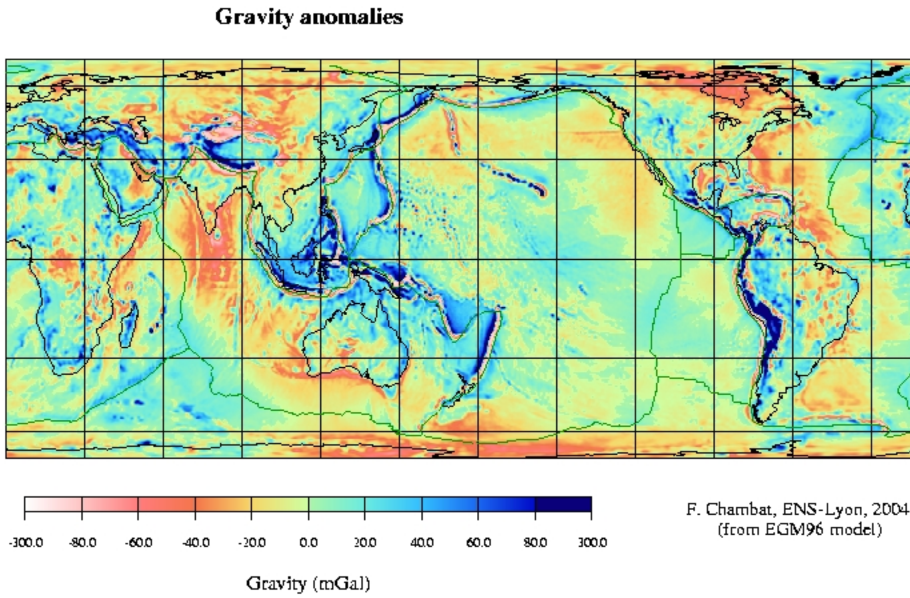
Bouguer anomaly (apply both free-air and Bouguer plate corrections):

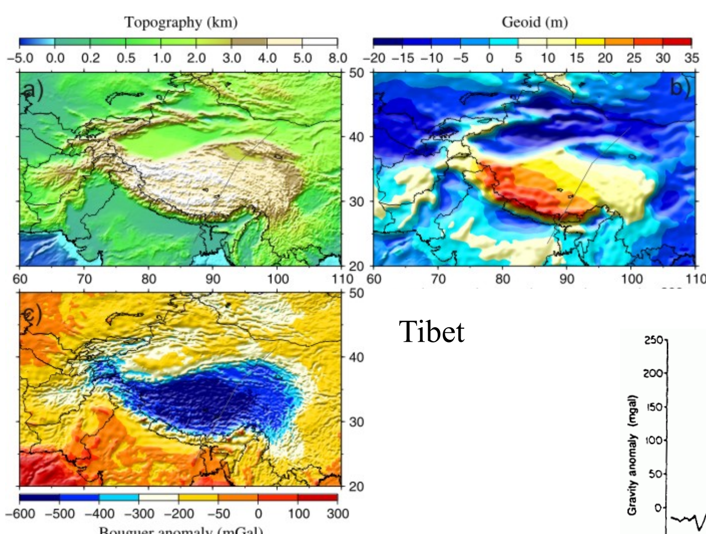
$\Delta g+\Delta g_{FA}-\Delta g_{BP} << 0$ because Bouguer removes additional mass.

Undercompenstated topography: A too-shallow root, yields $\Delta g+\Delta g_{FA}>0$

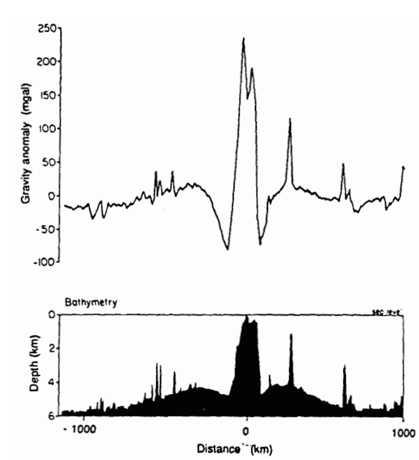
Overcompensated topography: A too-shallow root, yields $\Delta g+\Delta g_{FA}<0$

Free-Air Gravity Anomalies (global)





Tibet



Free-air gravity across the Hawaiian ridge